been connected with the University for nearly the entire period of its existence. One of the brilliant young men whom Professor Sylvester attracted to the University in its early days, he won straightway the favorable notice of that eminent man for the enthusiasm and intellectual acumen with which he entered upon the study of advanced mathematics, then almost an unknown science in this country ; and this fortunate combination of interest, energy, and ability characterized his entire career. At the time of his death he was occupied in the preparation of a treatise on the Theory of Surfaces. Undoubtedly the intense ardor with which he engaged in this work contributed in large measure to that impairment of the nervous system from which he had recently suffered. Professor Craig possessed great power of research, and wrote much for various mathematical journals. For many years he was editor of the *American Journal of Mathematics*, and it is largely due to his zeal and able direction that that journal continues to hold its high rank in the mathematical world. Professor Craig occupied a place in the very front rank of American mathematicians. His scientific ideals were the highest, and as teacher, editor, and investigator, he brought to his work a high degree of originality, and an intellectual ardor which was a source of inspiration to all with whom he was closely associated."

ON SYSTEMS OF ISOTHERMAL CURVES.*

By PROFESSOR L. E. DICKSON.

1. The object of this paper is to give an elementary geometrical definition of a system of isothermal curves in the plane. The definition is readily extended to families of curves on any algebraic surface. For simplicity of expression, the definition is given in connection with the two families of curves which are to be discussed at length ; the general definition will then be apparent. The usual methods of treating the subject are indicated in §§ 4-5.

2. The concentric circles about a point O have as orthogonal trajectories the straight lines through O. Select two of these lines, OP and OQ, and designate by q the number of radians in the angle POQ. On the circle about O with radius $OP=r$, the arc $PQ=qr$. On the line OP measure off from P the length $PT=qr$. [In Fig. 1, it is taken to the right of P]. On the circle about O with radius $OT=r+qr$, the arc TS equals $qr+q^2r$. Hence PQ, PT and QS are each of length qr, while the limit of $TS \div PT$ as q approaches zero is unity.

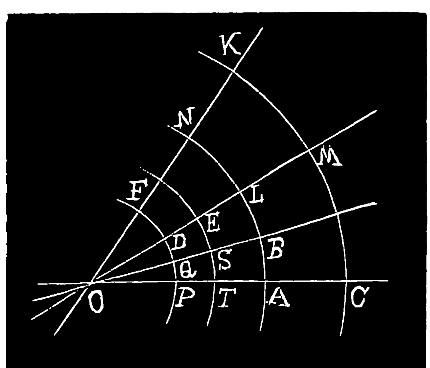

Fig. 1.

*Read before the American Association for the Advancement of Science.

Let L be an arbitrary point in the plane, but distinct from O. Denote by A and B the intersections of OP and OQ, respectively, with the circle about O with radius OL. Denote by D and E the intersections of OL with the circles through P and T. We define a curvilinear quadrilateral, one of whose vertices is L, as follows: Set $OA=a$, and denote by d the number of radians in the angle POD. Then arc $AB=qa$, $DE=qr$. On OP take $AC=AB=qa$; on the circle PQ take arc $DF=DE=qr$. The lines OD, OF and the circles about O with radii OA, OC intersect in the curvilinear quadrilateral $LNKM$. Its sides have the following lengths:

$$LM=qa, \quad LN=qa, \quad NK=qa, \quad MK=q(a+qa).$$

Hence, as q approaches zero, the limits of the ratios of the sides are all unity. The angles of the quadrilateral are always $\pi/2$. In the vicinity of an arbitrarily chosen point L, we obtain by this construction a network of rectangular curvilinear quadrilaterals, which tend to become squares as q approaches zero. The family of concentric circles and their orthogonal trajectories are therefore said to form a system of isothermal curves.

Instead of the particular "base lines" OP, OQ, circle PQ, and circle TS, we may employ any other pair of straight lines through O and pair of circles about O, such that the limits of the ratios of the quadrilateral formed are all unity.

3. Consider the family of circles $x^2+(y-r)^2=r^2$ tangent to the x-axis at the origin. In polar coördinates, their equations are

$$(1) \qquad \rho=2r\sin\theta \ (r=\text{parameter}).$$

Their orthogonal trajectories are the circles tangent to the y-axis at the origin; their equations are

$$(2) \quad \rho=2R\cos\theta \ (R=\text{parameter}).$$

We are to show that these curves form an isothermal system, according to the definition next given. Employing as base lines certain of the circles, PQ, TS, PT, and QS, we define a curvilinear quadrilateral $LNKM$, one of whose vertices L is an arbitrarily chosen point distinct from O, by making arc AB=arc AC, arc DF=arc DE, thereby determining the circles $OCMK$ and $OFNK$. If $PT=PQ$, then $LNKM$ tends to become a square as PQ approaches zero.

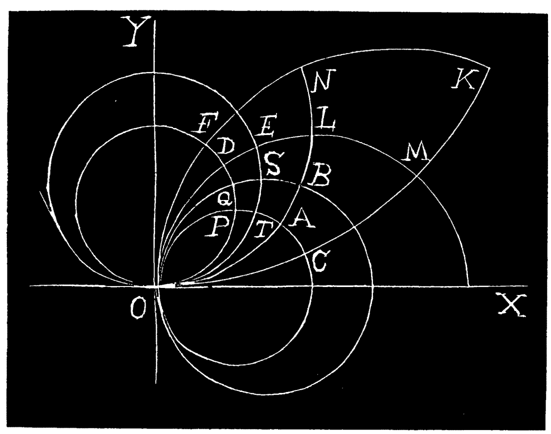

Fig. 2.

In proof, we consider the circles of unit radius

$$OPQ : \rho=2\sin\theta\ ; \quad OPT : \rho=2\cos\theta.$$

Their second point of intersection P has the coördinates

$$P : (\rho, \theta)\equiv(\sqrt{2}, \pi/4).$$

Let the point Q be determined on the circle OPQ so that the angle POQ shall contain q radians. Then arc $PQ=2q$. Take arc $PT=2q$ on the circle OPT. Since angle $XOQ=\frac{1}{4}\pi+q$, the length of OQ is

$$\rho=2\sin(\tfrac{1}{4}\pi+q)=\sqrt{2}(\cos q+\sin q)=\sqrt{2}(1-\frac{q^2}{2!}+\frac{q^4}{4!}-\ldots.+q-\frac{q^3}{3!}+\frac{q^5}{5!}-..).$$

Hence the coördinates of Q are

$$Q : \rho=\sqrt{2}(1+q+\ldots.). \quad \theta=\tfrac{1}{4}\pi+q,$$

where, as henceforth, the terms indicated by dots contain the factor q^2. By a like proof, or by symmetry, the coördinates of T are

$$T : \rho=\sqrt{2}(1+q+\ \ldots), \quad \theta=\tfrac{1}{4}\pi - q.$$

In order that a circle (2) shall contain Q, we find that

$$R=\frac{\sqrt{2}(1+q+\ldots.)}{2\cos(\frac{1}{4}\pi+q)}=\frac{1+q+\ldots.}{1-q+\ldots.}=1+2q+\ldots.$$

Hence the equation of the circle in question is

$$OQSB : \rho=2\cos\theta(1+2q+\ldots\ ).$$

Analogously, or by symmetry, the circle (1) through T is

$$OTSE : \rho=2\sin\theta(1+2q+\ldots.).$$

Consider an arbitrary point L. It may be determined as the intersection of the circle $OABL$ of set (1) with the circle $ODEL$ of set (2). These circles are determined by the points A and D, respectively. Let them have the coördinates

$$A : (\rho, \theta)\equiv(2\cos\alpha, \alpha)\ ; \quad D : (\rho, \theta)\equiv(2\sin\beta, \beta).$$

Then the circles $OABL$ and $ODEL$ have the respective equations

$$\rho=2\cot\alpha.\sin\theta, \quad \rho=2\tan\beta.\cos\theta.$$

The former intersects the circle $OQSB$ in the point B for which

$$\tan\theta_B = \tan\alpha.(1+2q+\ldots\ldots).$$

Denoting by σ the angle AOB, we have $\theta_B = \alpha+\sigma$, so that

$$\tan\alpha.(1+2q+\ldots\ldots) = \frac{\tan\alpha+\sigma+\frac{1}{3}\sigma^3+\ldots\ldots}{1-\tan\alpha(\sigma+\frac{1}{3}\sigma^3+\ldots\ldots)} = \tan\alpha+\sigma+\sigma\tan^2\alpha+\sigma^2(\quad).$$

Hence $\sigma = q\sin2\alpha+\ldots\ldots$ Since the radius of $OABL$ is $\cot\alpha$, we get

$$\text{arc}AC \equiv \text{arc}AB = 2\sigma\cot\alpha = 4q\cos^2\alpha+\ldots\ldots$$

Hence the point C has the coördinates

$$\rho = 2\cos\alpha+4q\sin\alpha\cos^2\alpha+\ldots\ldots,\quad \theta = \alpha-2q\cos^2\alpha+\ldots\ldots$$

In order that circle a (1) shall pass through C, we must have

$$r = \frac{\cos\alpha+2q\sin\alpha\cos^2\alpha+\ldots\ldots}{\sin(\alpha-2q\cos^2\alpha+\ldots\ldots)} = \frac{\cos\alpha(1+2q\sin\alpha\cos\alpha+\ \ldots)}{\sin\alpha-2q\cos^3\alpha+\ldots\ldots}$$

$$= \frac{\cos\alpha}{\sin\alpha}[1+2q(\sin\alpha\cos\alpha+\frac{\cos^3\alpha}{\sin\alpha})+\ldots\ldots] = \cot\alpha(1+2q\cot\alpha+\ldots\ldots).$$

The equation to the circle $OCMK$ is therefore

$$\rho = 2\sin\theta(\cot\alpha+2q\cot^2\alpha+\ldots\ldots).$$

For the intersection E of the circles $ODEL$ and $OTSE$, we have

$$\tan\theta_E = \tan\beta.(1-2q+\ldots\ldots),\quad \theta_E = \beta-q\sin2\beta+\ldots$$

$\therefore$ $\text{arc}DF \equiv \text{arc}DE = 4q\sin^2\beta+\ \ldots$
Hence the point F has the coördinates

$$\rho = 2\sin\beta+4q\cos\beta\sin^2\beta+\ldots\ldots,\quad \theta = \beta+2q\sin^2\beta+\ldots\ldots$$

The equation to the circle $OFNK$ of family (2) is therefore

$$\rho = 2\cos\theta(\tan\beta+2q\tan^2\beta+\ldots\ldots).$$

We may now determine the coördinates of L, M, N from the equations of the circles $OABL$, $ODEL$, $OCMK$, and $OFNK$:

$$\tan\theta_L = \tan\alpha\tan\beta,\quad \rho_L = 2(\tan^2\alpha+\cot^2\beta)^{-\frac{1}{2}},$$

$$\tan\theta_M = \tan\alpha\tan\beta - 2q\tan\beta + \ldots,$$

$$\tan\theta_N = \tan\alpha\tan\beta + 2q\tan\alpha\tan^2\beta + \ldots.$$

Denoting by τ the angle MOL, then $\theta_M = \theta_L - \tau$, whence

$$\tan\alpha\tan\beta - 2q\tan\beta + \ldots = \frac{\tan\theta_L - \tau - \frac{1}{3}\tau^3 \ldots}{1 + \tan\theta_L(\tau + \frac{1}{3}\tau^3 + \ldots)}$$

$$= \tan\alpha\tan\beta - \tau(1 + \tan^2\alpha\tan^2\beta) + \tau^2(\quad).$$

Since the radius of $ODEL$ is $\tan\beta$, we have

$$\mathrm{arc}\,ML = 2\tau\tan\beta = \frac{4q\tan^2\beta}{1 + \tan^2\alpha\tan^2\beta} + \ldots.$$

Determining the angle $LON \equiv \theta_N - \theta_L$, we find similarly that

$$\mathrm{arc}\,LN = 2\cot\alpha(LON) = \frac{4q\tan^2\beta}{1 + \tan^2\alpha\tan^2\beta} + \ldots.$$

Hence

$$\lim_{q \doteq 0} \frac{\mathrm{arc}\,ML}{\mathrm{arc}\,LN} = 1,$$

so that the families of circles (1) and (2) form an isothermal system. Also

$$\mathrm{arc}\,LN = q\rho_L{}^2 + \ldots.$$

$$\lim_{q=0} \frac{\mathrm{arc}\,LN}{\mathrm{arc}\,PQ} = \frac{2}{\tan^2\alpha + \cot^2\beta} = \tfrac{1}{2}\rho_L{}^2.$$

A curviliner quadrilateral, whose angles are all $\pi/2$ and the limits of the ratios of whose sides are all unity as q approaches zero, may be designated as an "infinitesimal square." We may state our result in the following terms :

The two families of circles (1) *and* (2) *form a network of infinitesimal squares whose sides are proportional to the squares of the distances from the origin to their nearest corner points.*

4. All systems of isothermal curves may be obtained by function-theory as follows : Set

$$X + iY = \theta(x + iy), \quad X - iY = \overline{\theta}(x - iy).$$

$$\therefore X = \tfrac{1}{2}[\theta(x + iy) + \overline{\theta}(x - iy)] \equiv U(x, y), \qquad Y \equiv \frac{-i}{2}(\theta - \overline{\theta}) \equiv V(x, y).$$

The lines $X = a$, $Y = b$ form an isothermal system. Under the transformation $X = U(x, y)$, $Y = V(x, y)$, the straight lines correspond to the curves $U(x, y) = a$, $V(x, y) = b$. The point (a, b) corresponds to the intersection P_1 of

$U=a$, $V=b$; the point $(a, b+q)$ corresponds to the intersection P_2 of $U=a$, $V=b+q$; the point $(a+q, b)$ corresponds to the intersection P_3 of $U=a+q$, $V=b$. It may be proved that angle $P_3P_1P_2=\frac{1}{2}\pi$, and that

$$\lim_{q\doteq 0} \frac{P_1P_2}{P_1P_3}=1.$$

To obtain the correspondence in which the sense of the angles is reversed, we set $X-iY=\theta(x+iy)$.

5. Isothermal systems may be treated from the standpoint of transformation-groups.* For the case of the orthogonal circles (§3), the system may be derived from the system of lines parallel to the axes by the familiar transformation through reciprocal radii vectors.† The latter transforms any isothermal system into an isothermal system since it leaves invariant the partial differential equation

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0,$$

satisfied by the function $U\equiv U(x, y)$ of §4.

The University of Chicago, February, 1901.

ATMOSPHERIC REFRACTION.

By G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The object of this article is not to set forth a new theory but simply to review the old. Let x=radius vector from center of the earth to any point in the path of the ray, φ=the angle between the ray and its normal, μ=the index of refraction, a, θ, μ_0=the values of x, φ, μ at the earth's surface, r=the atmospheric refraction.

Then $\mu x\sin\varphi=\mu_0 a\sin\theta$....(1).

Let φ'=the angle a consecutive element of the ray's path makes with the normal. Then $\varphi-\varphi'=dr$. By the laws of refraction

$$\mu\sin\varphi=(\mu+d\mu)\sin(\varphi-dr)=(\mu+d\mu)(\sin\varphi-dr\cos\varphi)$$

since $\sin dr=dr$ and $\cos dr=1$.

$\therefore\ d\mu\sin\varphi=\mu dr\cos\varphi$.

$\therefore\ dr=\dfrac{1}{\mu}\tan\varphi d\mu$, and $r=\displaystyle\int\frac{\tan\varphi d\mu}{\mu}$.

*Lie-Scheffers, *Differentialgleichungen*, pp. 156-157.

†Lie-Scheffers, *Geometrie der Beruhrungstransformationen*, pp. 6-9.